
Modern approaches to quantum gravity

Homework 5 Fall 2025

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1. Conserved quantitites and the spinning string

The Polyakov string is invariant under Poincaré transformations in the target space

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{1}$$

(a) Show, by working in conformal gauge, that the Noether currents associated to these global symmetries from the worldsheet are

$$P^\mu_\alpha = T\partial_\alpha X^\mu \quad \text{and} \quad J^{\mu\nu}_\alpha = P^\mu_\alpha X^\nu - P^\nu_\alpha X^\mu \tag{2}$$

where $T \equiv \frac{1}{2\pi\alpha'}$ is the string tension.

(b) Write down the conserved charge associated with Lorentz transformations in terms of the modes of the string, for both closed and open strings (with Neumann b.c.). What is the physical significance of the various contributions to the angular momentum?

(c) Consider a closed string that at time $t = 0$ is at rest and is a circle in the $x - y$ plane of radius R. Let $\sigma \in [0, 2\pi]$ at $\tau = t = 0$, be proportional to the arc length of the string:

$$x = R\cos(\sigma), \quad y = R\sin(\sigma) \tag{3}$$

At early times $\tau \ll 1$, the solution for the string modes can be written as:

$$X^0 = t = R\tau,\ X^1 = x + \mathcal{O}(\tau),\ X^2 = y + \mathcal{O}(\tau),\ X^i = 0,\ i > 2 \tag{4}$$

By computing the energy of the string, explain why T can be interpreted as the string tension.

(d) Now consider an open string ($\sigma \in [0, \pi]$) that is spinning in the $x - y$ plane, with

$$x = R\cos(\sigma)\cos(\tau), \quad y = R\cos(\sigma)\sin(\tau), \quad t = R\tau \tag{5}$$

Show that the energy and total angular momentum of this configuration are, respectively,

$$E = \pi RT, \quad J = \pi R^2 T/2 \tag{6}$$

Argue why this spinning string has the maximum amount of angular momentum per unit of energy squared achievable by a string, $\frac{J}{E^2} = \frac{1}{2\pi T} = \alpha'$. This ratio is called the *Regge slope*.

2. T-duality and D-branes

(a) Starting from the mode expansion of the bosonic closed string

$$\begin{aligned} X^\mu = x^\mu &+ \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma \\ &+ i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \left(\frac{\alpha_m^\mu}{m} e^{-im\sigma^+} + \frac{\tilde{\alpha}_m^\mu}{m} e^{-im\sigma^-} \right) \end{aligned} \tag{7}$$

where $\sigma^\pm = \tau \pm \sigma$, explain why, for non-compact directions X^μ, we impose $\alpha_0^\mu = \tilde{\alpha}_0^\mu$.

(b) Now consider the last direction X_{25} to be periodically identified with circumference $2\pi R$, i.e. $X_{25} \sim X_{25} + 2\pi R$. Argue that the condition mentioned aboved is weakened to

$$\sqrt{\frac{\alpha'}{2}}(\alpha_0^{25} - \tilde{\alpha}_0^{25}) = mR \qquad m \in \mathbb{Z} \tag{8}$$

and draw what string configurations $m = 0, -1, +1$ represent in the target space. m is called the winding mode of the corresponding string.

(c) Impose also that the string states are invariant under translation by $2\pi R$, which means $e^{2\pi R\, p^{25}} = 1$. Using that $p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$, show that this condition, together with the above one, implies

$$\alpha_0^{25} = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} + m\frac{R}{\alpha'} \right) \tag{9}$$

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}} \left(\frac{n}{R} - m\frac{R}{\alpha'} \right) \tag{10}$$

(d) Compute the mass spectrum, and argue that under the simultaneous exchange of $R \leftrightarrow \alpha'/R$ and $n \leftrightarrow m$, the mass spectrum is invariant. This is called T-duality. Note that its effect is to send $\tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25}$. We will also impose that it sends $\tilde{\alpha}_m^{25} \to -\tilde{\alpha}_m^{25}$ for all m.

(e) Now we will take a small detour before tackling the open string. Show that the mode expansion of the closed string X^μ (7) can be written as

$$X^{25}(\tau, \sigma) = X_L^{25}(\tau + \sigma) + X_R^{25}(\tau - \sigma) \tag{11}$$

and argue that under T-duality $X^{25} \to \hat{X}^{25}$, where

$$\hat{X}^{25}(\tau, \sigma) = X_L^{25}(\tau + \sigma) - X_R^{25}(\tau - \sigma) \tag{12}$$

(f) Consider an open string with Neumann boundary coditions and worldsheet coordinates (τ, σ), $\sigma \in [0, \pi)$. Write down its mode expansion and show that a linear term in σ is disallowed.

(g) Using the T-duality transformation $X^{25} = X_L^{25} + X_R^{25} \to \hat{X}^{25} = X_L^{25} - X_R^{25}$, find the T-dual mode expansion $\hat{X}^{25}$.

(h) The mode expansion was such that it obeyed the Neumann boundary condition $\partial_\sigma X^{25} = 0$ at $\sigma = 0, \pi$. Show that now,

$$\hat{X}^{25}(\pi) - \hat{X}^{25}(0) = 2\pi n \hat{R} \tag{13}$$

where $\hat{R} = \alpha'/R$ is the T-dual radius. You just proved that under T-duality, Neumann boundary conditions are interchanged with Dirichlet boundary conditions. Draw in target space what string configurations $n = 0, 1$ correspond to.

(i) Starting from our preferential Neumann boundary conditions and using T-duality, argue that there exists a limit allowing to obtain Dirichlet boundary conditions even for uncompactified X^{25}.